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Space-charge-limited current flow between plane-parallel electrodes in a low-density gas

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Abstract. The electrons produced by cathode emission in a plane diode generate ions by collisions with background gas atoms. Poisson's equation is solved numerically in the steady state for non-relativistic particle motion assuming that, at the cathode surface, there is zero field and an abundant supply of zero-energy electrons. A suitable choice of non-dimensional variables enables the interelectrode potential distribution, space charge distribution and electron current density to be presented quite generally as a function of the gas filling parameters. The calculations are carried out for anode potentials up to 30 kV and for diode currents up to 1·7 times the Child–Langmuir vacuum limit. Depletion of neutral particles defines two modes of diode operation to which these calculations are applicable. The first is the pulsed mode on a time scale over which the neutral depletion is negligible and the second mode is the final steady state in which the ion flux is balanced by an opposing self-diffusion flux of neutral particles. Finally, the calculations are applied to diodes with a xenon gas filling and it is shown that the above currents can be generated with less than 1% gas scattering of the electron beam.

1. Introduction

Electron beam generators are usually designed to provide the maximum possible beam current for a given applied voltage and electrode geometry. If the cathode emits sufficient electrons then this ultimate current occurs when the field at the cathode surface is reduced to zero by the effects of negative space charge. This is approximately true even when the cathode emits by field emission since, at very high powers, a plasma quickly forms on the face of the cathode electrode. The plasma subsequently acts as a field emitting cathode of very low work function and copious emission takes place for very low values of cathode field. In some electron beam applications it is not permissible to achieve a substantial increase in beam current by raising the applied voltage since the electrons are required to have a prescribed energy. An increase in cathode emitting area or decrease in electrode separation will increase the total current, but there may be experimental factors that prevent such modification. A current increase may be achieved more simply by reducing the negative space charge in the cathode region through the introduction of positive ions. Wheeler (1974a) discussed how this can be done in a plane electrode configuration by injecting ions normally through the cathode and allowing them to reflex back again. However, very powerful ion sources would be required and a more convenient experimental procedure is to inject neutral particles into the diode and produce the ions internally by electron impact. Wheeler (1974b) has evaluated the electron currents attainable in this situation as a function of the position and density of a thin sheet of gas atoms injected parallel to the electrodes. This paper considers the very simple experimental arrangement of a diode that has a low density, uniform gas filling. The theoretical analysis is more complex than the earlier neutral sheet problem in that the basic differential equation is raised from the second to the third order. Introduction of a gas within the diode naturally produces a scatter in the electron beam. However, the beam–gas interaction required by the theory is very small since one slow ion can neutralise many fast electrons.

2. Mathematical formulation

Consider a plane-parallel electrode geometry comprising a cathode at zero potential and an anode at potential V_a distant a away. Let z measure the spacial distance normally from the cathode towards the anode and let n be the number density of the gas filling atoms. If $J_-(z')$ is the electron current density at position z' then between planes at z' and $z'+\mathrm{d}z'$ there are $(n/e)J_-(z')\sigma_-(z')\,\mathrm{d}z'$ ionising electron–atom collisions per unit area per second where $\sigma_-(z')$ is the atom cross section for single ionisation appropriate to the electron energy at position z'. The influence of other ionising collisions produced by electron impact is considered in § 2.3. One additional electron is produced at each collision and, in the steady state, the total electron current density at position z is obtained by integrating over $0 \leq z' \leq z$:

$$J_-(z)=J_-(0)\exp\left[n\int_0^z \sigma_-(z')\,\mathrm{d}z'\right] \tag{1}$$

where $J_-(0)$ is the current density at the cathode surface. In order to evaluate the negative space charge density at any position it is necessary to express $J_-(z)$ in its two principal velocity groups of electrons:

$$J_-(z)=J_-(0)\exp\left[-n\int_0^z \sigma_-(z')\,\mathrm{d}z'\right] + J_-(0)\left\{\exp\left[n\int_0^z \sigma_-(z')\,\mathrm{d}z'\right]-\exp\left[-n\int_0^z \sigma_-(z')\,\mathrm{d}z'\right]\right\}. \tag{2}$$

The first term here is the contribution due to electrons that leave the cathode and reach the position z without suffering a collision. These electrons are freely accelerated through the potential $V(z)$. The second term represents pairs of electrons starting at a collision at position z' and subsequently freely accelerated through the potential $V(z)-V(z')$. It is assumed that, after impact, the ionising electron is at rest and also that the electron produced by ionisation is initially at rest. If the probability of an electron suffering a collision between the electrodes is small then the exponentials may be expanded and only first-order terms in the collision integral need be retained. Division by the appropriate electron velocity then enables the negative space charge density to be expressed:

$$\rho_-(z)=-\left(\frac{m}{2e}\right)^{1/2}J_-(0)\left\{V^{-1/2}(z)\left[1-n\int_0^z \sigma_-(z')\,\mathrm{d}z'\right] + 2n\int_0^z \sigma_-(z')[V(z)-V(z')]^{-1/2}\,\mathrm{d}z'\right\}. \tag{3}$$

This expression is valid providing there is no potential maximum between the electrodes where slow electrons might accumulate, and all potentials must be sufficiently low for non-relativistic treatment of electron motion to apply. Furthermore, it is assumed that the electrons travel normally from cathode to anode; this implies that the magnetic field produced by the current flow is insufficient to perturb significantly the straight line trajectories. Ions produced by electron impact are accelerated from rest towards the cathode. However, the probability of collision with a neutral atom is significant since the cross section for symmetrical charge transfer is large. These latter collisions produce no additional ions and, in the steady state, the total ion current density at any position z is obtained by integration over all ionising collisions between position z and the anode:

$$J_+(z) = n\int_z^a J_-(z')\sigma_-(z')\,\mathrm{d}z'. \tag{4}$$

In order to evaluate the positive space-charge density this current density must be expressed in its principal ion velocity groups:

$$J_+(z) = n\int_z^a J_-(z')\sigma_-(z')\exp\left[-n\int_z^{z'} \sigma_+(z', z'')\,\mathrm{d}z''\right]\mathrm{d}z' + n\int_z^a J_-(z')\sigma_-(z')\left\{1-\exp\left[-n\int_z^{z'} \sigma_+(z', z'')\,\mathrm{d}z''\right]\right\}\mathrm{d}z'. \tag{5}$$

The first term here is the contribution due to ions produced at position z', where $z < z' < a$, that are accelerated through the potential $V(z') - V(z)$ to reach position z without suffering a charge transfer collision. $\sigma_+(z', z'')$ is the symmetrical charge transfer cross section for an ion of energy $V(z') - V(z'')$, where $z < z'' < z'$. The second term represents ions that are produced by charge transfer at position z'' and then freely accelerated through the potential $V(z'') - V(z)$. Substitution for $J_-(z')$ from equation (1), expansion of the exponentials to first order in the collision integrals and division by the appropriate ion velocity then enables the positive space-charge density to be expressed:

$$\rho_+(z) = \left(\frac{M}{2e}\right)^{1/2} J_-(0)n\left\{\int_z^a \sigma_-(z')[V(z') - V(z)]^{-1/2} \times\left[1 - n\int_z^{z'} \sigma_+(z', z'')\,\mathrm{d}z'' + n\int_0^{z'} \sigma_-(z'')\,\mathrm{d}z''\right]\mathrm{d}z' + n\int_z^a \sigma_-(z')\int_z^{z'} \sigma_+(z', z'')[V(z'') - V(z)]^{-1/2}\,\mathrm{d}z''\,\mathrm{d}z'\right\}. \tag{6}$$

This expression assumes that there is no potential minimum between the electrodes where slow ions might accumulate and also that the ions produced by charge transfer are initially at rest.

2.1. *Poisson's equation*

$$\mathrm{d}^2V(z)/\mathrm{d}z^2 = -4\pi[\rho_+(z) + \rho_-(z)]. \tag{7}$$

A first integration with respect to z can be performed after multiplying both sides of this equation by $2\mathrm{d}V(z)/\mathrm{d}z$. It is then convenient to take potential as the variable of integration instead of the spacial variable. This is achieved by putting $dz' = dV(z')[dV(z')/dz']^{-1}$. A great simplification follows if dimensionless potential and spacial variables are introduced:

$$x = z/a \qquad y = V(z)/V_a \tag{8}$$

and also if the cathode current density $J_-(0)$ is expressed as a multiple of the Child–Langmuir space-charge-limited electron current density, J_0, for the vacuum diode:

$$J_0 = \frac{1}{9\pi}(2e/m)^{1/2} V_a^{3/2} a^{-2}. \tag{9}$$

Equations (3), (6), (7), (8) and (9) then lead to the following equation for the fully space-charge-limited current $J_-(0)$, corresponding to zero cathode field.

$$\begin{aligned}
\frac{9}{16}\frac{J_0}{J_-(0)}\left(\frac{dy}{dx}\right)^2 = {} & y^{1/2} - \left(\frac{M}{m}\right)^{1/2}\frac{na}{2}\int_0^y\int_y^1 \frac{\sigma_-(y')}{(dy'/dx)(y'-y)^{1/2}}\,dy'\,dy \\
& - \left(\frac{M}{m}\right)^{1/2}\frac{n^2a^2}{2}\left(\int_0^y\int_y^1 \frac{\sigma_-(y')}{(dy'/dx)}\int_y^{y'} \frac{\sigma_+(y',y'')}{(dy''/dx)(y''-y)^{1/2}}\,dy''\,dy'\,dy\right. \\
& - \int_0^y\int_y^1 \frac{\sigma_-(y')}{(dy'/dx)(y'-y)^{1/2}}\int_y^{y'} \frac{\sigma_+(y',y'')}{(dy''/dx)}\,dy''\,dy'\,dy \\
& \left. + \int_0^y\int_y^1 \frac{\sigma_-(y')}{(dy'/dx)(y'-y)^{1/2}}\int_0^{y'} \frac{\sigma_-(y'')}{(dy''/dx)}\,dy''\,dy'\,dy\right) \\
& + \frac{na}{2}\left(2\int_0^y\int_0^y \frac{\sigma_-(y')}{(dy'/dx)(y-y')^{1/2}}\,dy'\,dy - \int_0^y y^{-1/2}\int_0^y \frac{\sigma_-(y')}{(dy'/dx)}\,dy'\,dy\right).
\end{aligned} \tag{10}$$

The four terms comprising the right-hand side of this equation are presented in order of significance and the space charges responsible have their respective origins in:

(i) the electrons, in absence of ionising collisions,
(ii) the ions, in absence of charge transfer collisions,
(iii) the slower ions resulting from charge transfer collisions,
(iv) the slower electrons resulting from ionising collisions.

For most gas atoms the cross section for ionisation by electron impact is well represented by the two-parameter Bethe–Born relation:

$$\sigma_-(V) = \frac{B}{V}\ln(CV) \tag{11}$$

where V is the potential energy of the impacting electron, assumed non-relativistic. This relation indicates a threshold at $V = C^{-1}$ and a maximum cross section at $V = 2{\cdot}72C^{-1}$. In terms of the parameter y the cross section becomes

$$\sigma_-(y) = \sigma_-(V_a)\frac{\ln(Dy)}{y\ln D} \qquad \text{where } D = CV_a. \tag{12}$$

2.2. *Numerical solution*

If the gas filling density is sufficiently small then the third and fourth terms on the right-hand side of equation (10) are negligible in comparison to the first two. The technique of solution adopted is iterative, starting with the simple vacuum relation, i.e. $n = 0$ in equation (10), and subsequently increasing the filling density by small increments. A correction due to the effects of slower ions and slower electrons is considered in § 2.3. With these simplifications and substitutions equation (10) can be written:

$$\frac{9}{16}\frac{J_0}{J_-(0)}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^{1/2} - A\int_0^y\int_y^1\left[y'(y'-y)^{1/2}\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)\ln D\right]^{-1}\ln(Dy')\,\mathrm{d}y'\,\mathrm{d}y \qquad (13)$$

where

$$A = \frac{na}{2}\left(\frac{M}{m}\right)^{1/2}\sigma_-(V_a). \qquad (14)$$

It is now convenient to assume that $\mathrm{d}y'/\mathrm{d}x$ in the integrand above is a known function of y'. The equation can then be regarded as separable and direct integration gives

$$x\left(\frac{J_-(0)}{J_0}\right)^{1/2} = \frac{3}{4}\int_0^y\left\{y^{1/2} - A\int_0^y\int_y^1\left[y'(y'-y)^{1/2}\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)\ln D\right]^{-1}\ln(Dy')\,\mathrm{d}y'\,\mathrm{d}y\right\}^{-1/2}\mathrm{d}y. \qquad (15)$$

It must be remembered that y' here has the minimum value of D^{-1} (threshold); otherwise the cross section $\sigma_-(y')$ will be negative. This equation is solved numerically and iteratively starting with a trial expression for $\mathrm{d}y'/\mathrm{d}x$ in terms of y'. For small values of A this initial approximation is the Child–Langmuir law, corresponding to $A = 0$ and $J_-(0) = J_0$ in equation (15). In this case integration leads to $x = y^{3/4}$, giving $\mathrm{d}y/\mathrm{d}x = \frac{4}{3}y^{1/4}$. Numerical evaluation of the three successive integrals in equation (15) then gives a first estimate of x as a function of y and $J_-(0)/J_0$. In particular the coordinate $x = 1$, $y = 1$ determines the value of $J_-(0)/J_0$ for the values of A and D chosen. Substitution of $J_-(0)/J_0$ into equation (13) then gives improved values of $\mathrm{d}y'/\mathrm{d}x$ which in turn, by substitution into equation (15), gives an improved value of $J_-(0)/J_0$. The integrands frequently diverge to infinity but the integrals are always finite. In these divergent regions the integrands are approximated by interpolated power series and the integrations performed analytically. For each successive value of A the trial expression taken for $\mathrm{d}y'/\mathrm{d}x$ is that appropriate to the preceding smaller value of A, corresponding to the same value of D.

Equations (12) and (14) show that the parameters D and A depend on both the nature of the filling gas and the anode potential. A range $40 < D < 4000$ is chosen which, for a threshold potential of 25 V, corresponds to $1\ \mathrm{kV} < V_a < 100\ \mathrm{kV}$. The larger values of D require a greater number of iterative cycles and smaller increments in the value of A. Calculations are terminated at $J_-(0)/J_0 = 1{\cdot}7$ which requires four iterative cycles of integration and increments of 5×10^{-3} in A to reproduce the current to $<0{\cdot}2\%$ at $D = 4000$. Figure 1 shows the value of the parameter A required to obtain a particular multiple of the Child–Langmuir current for $D = 40$, 400 and 4000. The strong convergence of the iterative technique employed is due to the small departure of the potential distribution from that appropriate to the Child–Langmuir law ($A = 0$). This is shown in figure 2 which is a plot of equation (15) for $J_-(0)/J_0 = 1{\cdot}7$ and for $D = 40$ and 4000. On the scale of this diagram the plot for $D = 400$ would not be resolved from the central curve $x = y^{3/4}$, corresponding to $A = 0$. It is apparent that there are no potential maxima or minima between the electrodes, as assumed when formulating the space charge densities. Equations (6), (8) and (9) enable the positive space-charge density to be expressed in non-dimensional form and figure 3 shows the distribution for $J_-(0)/J_0 = 1{\cdot}7$ and for $D = 40, 400$ and 4000. This density is zero at the

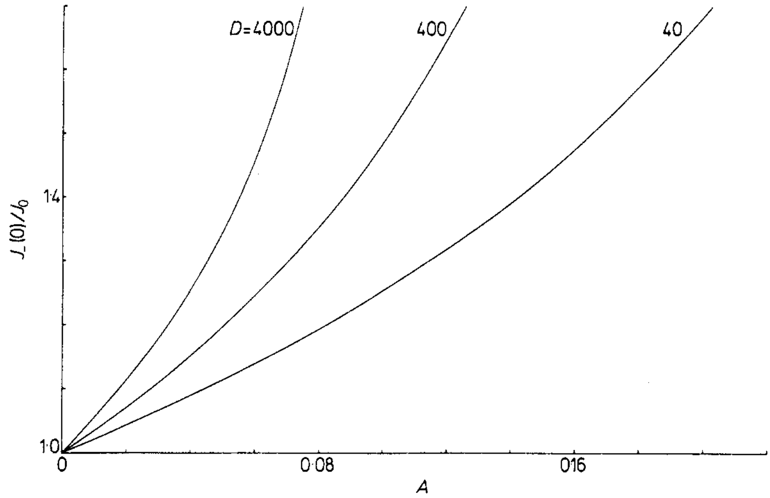

Figure 1. The electron current density as a function of the gas filling parameters A and D.

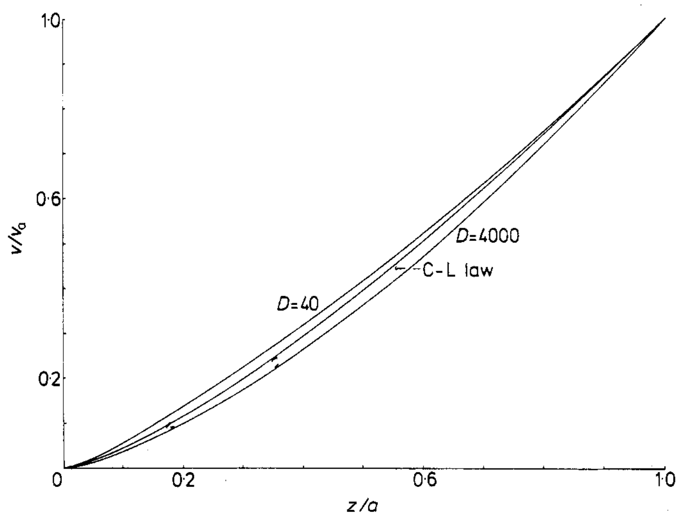

Figure 2. The interelectrode potential distribution for $J_-(0)/J_0 = 1{\cdot}7$ and for selected values of D.

anode, since ions cannot drift towards that electrode, and the density increases as the cathode is approached. However, there is a maximum of positive charge just in front of the cathode surface ($y = 0$). This arises because ionisation by electron impact cannot take place in the region $0 < y < D^{-1}$; therefore any ion reaching the cathode does so with a velocity appropriate to an energy of at least the threshold potential. On the other hand, in the interelectrode region, $D^{-1} < y < 1$, there are always some ions instantaneously at rest which contribute significantly to the local positive space charge. For $D = 400$ and 4000 the maxima are too close to the cathode to be resolved in the figure, but for $D = 40$ the maximum is apparent and reference to figure 2 shows that it is located in the vicinity of $y = 2{\cdot}72/D$ where, from equation (12), the ionisation cross section maximises.

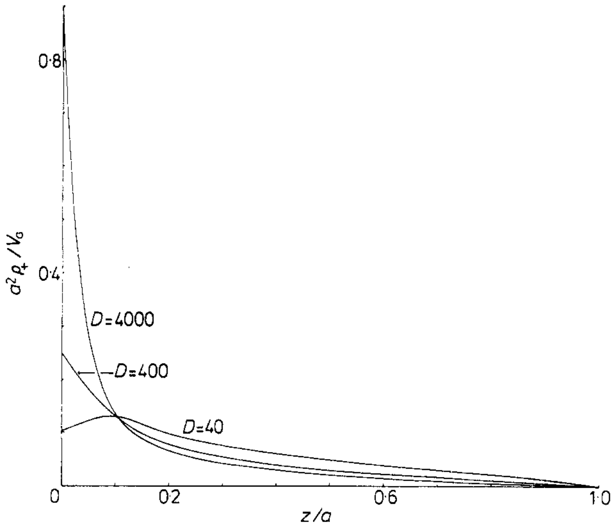

Figure 3. The positive space-charge distribution for $J_-(0)/J_0 = 1{\cdot}7$ and for selected values of D.

2.3. Limitations to the theory

The influence of multiply charged ions, produced by electron–neutral impact, can be included in the analysis by taking the appropriate sum over partial cross sections. These partial cross sections are well represented by equation (11), which is generalised to

$$\sigma_-^z(V) = \frac{B_z}{V} \ln(C_z V) \tag{16}$$

where z is here the charge of the ion produced. The collision term that dominates the space-charge distribution is the first term on the right-hand side of equation (6). Addition of the terms for multiply charged ions into the integral shows that their combined effect is equivalent to defining the cross section as

$$\sigma_- = \sum_{z=1} z^{1/2} \sigma_-^z. \tag{17}$$

It readily follows from equations (16) and (17) that the simple expression of equation (11) can be retained, provided B and C are defined as:

$$B = \sum_{z=1} z^{1/2} B_z \qquad C = \prod_{z=1} (C_z)^{z^{1/2}B_z/B}. \tag{18}$$

It should be noted that these are not the same parameters that define the more usual gross cross section. In this latter case it is ion current, not space charge, that is under consideration and this leads to $z^{1/2}$ replaced by z in equation (18).

Collisions between electrons and singly charged ions must also be considered since the ion has a cross section for further ionisation by electron impact that is comparable to that of the neutral atom. An estimate of the effect of these collisions is determined by assuming that the ion density n_+ is uniform across the interelectrode gap. If the electron–ion cross section $\sigma_-^+(V)$ has the same relative energy dependence as $\sigma_-(V)$

then the parameter A of equation (14) is increased by the following amount:

$$A \rightarrow [1+\sqrt{2} n_+ \sigma_-^+(V_a)/n\sigma_-(V_a)]A. \quad (19)$$

The ion density distribution $n_+(z)$ is proportional to the positive space-charge density $\rho_+(z)$, and if $Y(z)$ denotes the ordinate of figure 3, then

$$n_+(z) = Y(z) V_a/ea^2. \quad (20)$$

By taking the maximum value of $n_+(z)$ an upper limit to the increase in A can be found.

Higher-order terms in the exponential expansion of equation (1) influence both $\rho_-(z)$ and $\rho_+(z)$. However, the effect on $\rho_-(z)$ can be disregarded since it will be shown that even the first-order term, namely the slow electron term, has a negligible influence on the cathode current. The next significant term in $\rho_+(z)$, which arises from the first-order term in equation (1) via equation (4), is more significant and its influence on equation (13) is to increase the parameter A by an amount always less than

$$A \rightarrow (1+P_-)A$$

where

$$P_- = n \int_0^a \sigma_-(z) . \mathrm{d}z.$$

Physically P_- is the probability that an electron suffers an ionising collision with a neutral particle in its passage across the interelectrode gap. Elimination of n through equation (14) enables P_- to be expressed as

$$P_- = 2\left(\frac{m}{M}\right)^{1/2} \frac{A}{\sigma_-(V_a)} \int_0^1 \sigma_-(x)\, \mathrm{d}x.$$

The integral here can be evaluated analytically in terms of $\sigma_-(V_a)$ using equation (11) and assuming that the potential distribution obeys the Child–Langmuir law. For $40 < D < 4000$ this calculation shows that $(A/\sigma_-(V_a)) \int_0^1 \sigma_-(x)\, \mathrm{d}x$ maximises within the range of D and for $J_-(0)/J_0 = 1{\cdot}7$ it has the peak value of 0·50. Therefore within the range of the present calculations

$$P_- \leqslant 1{\cdot}0(m/M)^{1/2}. \quad (21)$$

Consequently the higher-order terms cannot increase the parameter A by more than the following amount:

$$A \rightarrow [1+(m/M)^{1/2}]A. \quad (22)$$

At sufficiently high anode potentials the proper relativistic treatment of the electron motion leads to space-charge densities greater than those given by equations (3) and (6). If only first-order terms in the parameter eV/mc^2 are retained in the expression for the electron velocity then the principal negative space-charge term in equation (10) is modified as follows:

$$y^{1/2} \rightarrow (1+0{\cdot}25 y e V_a/mc^2) y^{1/2}.$$

In order to assess the significance of the term in brackets the whole expression is averaged over the electrode gap, assuming a potential distribution obeying the Child–Langmuir law. The average value of $y^{1/2}$ then undergoes the modification:

$$\overline{y^{1/2}} \rightarrow (1+0{\cdot}139 e V_a/mc^2) \overline{y^{1/2}}. \quad (23)$$

The principal positive space-charge term in equation (10) is modified, by relativistic effects, through the parameter $\sigma_-(y)$. A first-order connection to the ionisation cross section formula, equation (11), can be evaluated from the general expression given by Mott and Massey (1965):

$$\sigma_-(y) \rightarrow \{1 + 1{\cdot}5y[1 - 1/\ln(Dy)]eV_a/mc^2\}\sigma_-(y).$$

The relativistic correction increases with the value of D, since $V_a = D/C$, and for $D = 4000$ the averaging process gives

$$\overline{\sigma_-}(y) \rightarrow (1 + 0{\cdot}153eV_a/mc^2)\overline{\sigma_-}(y). \tag{24}$$

The proximity of the two numerical factors in equations (23) and (24) suggests that a good first-order relativistic correction to equation (13) is simply a multiplicative factor of $(1 + 0{\cdot}146eV_a/mc^2)$ on the right-hand side. This implies that the current ratio previously calculated undergoes the following modification:

$$J_-(0)J_0 \rightarrow (1 - 0{\cdot}146eV_a/mc^2)J_-(0)/J_0. \tag{25}$$

The self magnetic field of the current flow tends to pinch the electron beam, producing a smaller beam diameter at the anode than at the cathode. A simple analysis of the electron trajectory, assuming uniform current density and a constant electron energy of $\frac{1}{2}eV_a$, shows that electrons emitted from the cathode at radius r from the centre of symmetry reach the anode at radius

$$[1 - 0{\cdot}157(eV_a/mc^2)J_-(0)/J_0]r.$$

This pinching has less effect on the cathode negative space charge than on the cathode positive space charge since $\rho_-(0)$ is generated by electrons directly emitted from the cathode whereas $\rho_+(0)$ is generated by ions that are produced throughout the gap, and which subsequently drift towards the cathode. A gross overestimate of the effect on the net current can be deduced by assuming that $\rho_-(z)$ everywhere is unchanged by pinching, whereas $\rho_+(z)$ everywhere is increased by the amount appropriate to the pinched electron current density at the anode. This implies that $J_-(z')$ in equation (4) is increased to the value

$$J_-(0)[1 + 0{\cdot}314(eV_a/mc^2)J_-(0)J_0].$$

If the analysis of § 2 is pursued with this modification it is found that the parameter A undergoes a similar increase. Consequently the influence of the self magnetic field is to increase the parameter A by an amount less than

$$A \rightarrow [1 + 0{\cdot}314(eV_a/mc^2)J_-(0)/J_0]A. \tag{26}$$

According to Massey and Gilbody (1974) the cross section of an ion for symmetrical charge transfer has the following dependence on energy:

$$\sigma_+ = \alpha(\beta - \ln V)^2$$

providing the potential V is well above threshold. Consequently σ_+ falls off with increase of potential far more slowly than σ_- of equation (11). This implies that the effects of charge transfer collisions, neglected in § 2.2, become more important as the anode voltage is increased. The third term on the right-hand side of equation (10) represents the influence of these collisions and the comparatively weak dependence of

σ_+ on potential suggests that an average cross section can be taken outside the integral without much loss of accuracy. If this is done then, in conjunction with equation (12), the parameter outside the bracket is $Ana\bar{\sigma}_+ = AP_+$, where P_+ is the probability that an ion suffers such a neutral collision in traversing the electrode gap. For a prescribed value of A the value of P_+ increases with increase in anode voltage simply because, from equations (11) and (14), na is almost proportional to V_a. Numerical solution of equation (10) for $0 \leq P_+ \leq 0{\cdot}15$ shows that, at $J_-(0)/J_0 = 1{\cdot}7$, the cathode current is increased by 1% for each increment of 5% in the probability P_+.

The influence of slow electrons, resulting from ionising collisions between fast electrons and neutral particles, is given by the fourth term on the right-hand side of equation (10). If σ_- is substituted from equation (12) then the parameter outside the bracket here is $A(m/M)^{1/2}$. Numerical solution of the equation, for $J_-(0)/J_0 = 1{\cdot}7$, and with an ion mass equal to that of the proton, shows that the slow electrons cause a reduction in cathode current of less than 0·1%.

2.4. *Establishment of the steady state*

If the neutral background density remains uniform and constant, as assumed in the previous subsections, then the steady state is established when the first ion produced near the anode has drifted to the cathode region. This concept of the steady state cannot persist since the production of ions necessitates the depletion of neutral particles. However, it is possible to define a time regime over which the foregoing theory is valid within a specified accuracy. This regime is limited at its lower end by the above ion transit time and the upper limit is defined to be the time taken for neutral depletion to cause a current reduction of less than 2%. As time proceeds the concentration of neutral particles at the cathode builds up due to neutralisation of ions at the electrode surface. Subsequently these neutral particles are transported away from the cathode by self-diffusion. Eventually a steady state is reached when, across all planes, the net flux of positive ions is balanced by an opposing diffusion flux of neutral particles. The neutral-particle density distribution across the electrode gap can be calculated in this situation and conditions established under which the deviation from the initial filling density is sufficient to cause less than 2% change in the diode current.

The transit time of an ion from the anode, where it is produced at rest, to the cathode in a potential distribution defined by the Child–Langmuir law is

$$\tau_1 = 1{\cdot}07a(M/eV_a)^{1/2}. \tag{27}$$

The neutral density begins to deplete everywhere as soon as diode current flows and the decay time constant at any coordinate z is

$$\tau(z) = e/\sigma_-(z)J_-(z).$$

This decay is fastest at the coordinate where the cross section has its maximum value σ_M. If the neutral density at this coordinate falls by 2% then the effect on the diode current is less than if the neutral density everywhere fell by 2%. In this latter case figure 1 shows that the associated reduction of A to $0{\cdot}98A$ produces a fall in $J_-(0)/J_0$ that is less than 2%. Consequently a time scale τ_2 can be defined over which the current ratio falls certainly by less than 2%:

$$\tau_2 = 0{\cdot}02e/\sigma_M J_-(0).$$

Elimination of $J_-(0)$ through the current ratio and equation (9) leads to

$$\tau_2 = 0{\cdot}400(me)^{1/2}\frac{a^2}{\sigma_M V_a^{3/2}} J_0/J_-(0). \tag{28}$$

The time scales τ_1 and τ_2 define a time interval for pulsed operation of the diode throughout which the current ratio departs by less than 2% from the values presented in figure 1.

In the final steady state the concentration gradient of neutral density is determined by the local balance between ion flux and diffusion flux:

$$d\, \partial n(z)/\partial z = (1/e)J_+(z),$$

where d is the coefficient of self-diffusion. This equation can be integrated analytically from cathode to anode after substituting for $J_+(z)$ from equations (1), (4), (9), (12) and (14) and assuming a Child–Langmuir potential distribution. In particular, the neutral density at the cathode $n(0)$ is increased above the initial filling density n:

$$n(0) = n[1 + F\gamma(D)] \tag{29}$$

where $0{\cdot}155 \leqslant \gamma(D) \leqslant 0{\cdot}286$ for $40 \leqslant D \leqslant 4000$. Also the density at the anode $n(a)$ is decreased:

$$n(a) = n[1 - F\delta(D)] \tag{30}$$

where $0{\cdot}116 \geqslant \delta(D) \geqslant 0{\cdot}095$ for $40 \leqslant D \leqslant 4000$. In these two equations:

$$F = \frac{2}{3\pi}\frac{A}{af}\left(\frac{2}{Me}\right)^{1/2} V_a^{3/2} J_-(0)/J_0$$

where the dependence of the diffusion coefficient on density has been removed by putting $d = f/n$. The diode current is more sensitive to ion production in the cathode region than in the anode region; moreover $\gamma(D) > \delta(D)$ for all D. These two observations imply that the condition $F\gamma(D) \leqslant 0{\cdot}02$ is sufficient to ensure that, in the steady state, the diode current is less than 2% greater than the value given by figure 1. This condition can be interpreted as an upper limit to anode potential:

$$V_a^{3/2} \leqslant 6{\cdot}66 \times 10^{-2}\frac{af}{A}\frac{(Me)^{1/2}}{\gamma(D)} J_0/J_-(0). \tag{31}$$

The time scale to establish this steady state is determined by the competing processes of neutral depletion and neutral diffusion. Neutral depletion is slowest at the anode where the electrons have their greatest energy and so experience the smallest ionisation cross section. The decay time constant here is

$$\tau(a) = e/\sigma_-(V_a)J_-(a).$$

Equations (9) and (11) enable this to be expressed as

$$\tau(a) = \frac{20(me)^{1/2}a^2}{V_a^{1/2} B \ln(CV_a)} J_0/J_-(0).$$

The time constant of the diffusion process is determined by the decay constants associated with the spacial Fourier components of the neutral density build up at the cathode surface. The lowest harmonic has the greatest time constant, which is

$$\tau(1) = a^2/\pi^2 d = na^2/\pi^2 f.$$

Over the range of variables to which the calculations have been applied it is found that $\tau(a) \gg \tau(1)$. Consequently the time scale to establish the steady state, within the restriction of equation (31), is certainly less than the time τ_3 required for the anode region to deplete by 2% in neutral density:

$$\tau_3 = 0{\cdot}02\tau(a). \tag{32}$$

3. Application

Figure 1 shows that the value of A required to produce a prescribed value of $J_-(0)/J_0$ has a weak dependence on the parameter D. This permits a good interpolation formula to be constructed giving A as a quadratic function of $\lg D$, thus enabling A to be evaluated at selected values of V_a where $D = CV_a$ and C is a function of the filling gas. For example, at $J_-(0)/J_0 = 1{\cdot}7$ interpolation gives

$$A = 0{\cdot}0745 + 0{\cdot}0387 \lg(4000/D) + 0{\cdot}0126[\lg(4000/D)]^2.$$

Equation (14) infers that massive atoms of large ionisation cross section, such as mercury or the heavy alkalies, are most effective in reducing the negative space charge. However, the cathode emission process can be greatly impaired by the presence of reactive vapours and an inert gas filling is far more acceptable in practice. Xenon is chosen for the purpose of illustration since it has both a large mass and cross section, therefore yielding a lower filling density than the lighter inert gases. In addition, the value of $P_+ = na\bar{\sigma}_+$ is lower, so minimising the correction to be applied for the effects of charge transfer collisions. Furthermore, equation (22) shows, for xenon, that the correction due to higher-order terms in the expansion is $A \to 1{\cdot}002A$ which, to the accuracy representable in figure 1, is a negligible correction. Schram *et al* (1965) and Schram (1966) have measured the partial ionisation cross sections of the inert gases at electron energies up to 20 keV and found excellent agreement with the theoretical relation (16). In conjunction with equation (18) these partial cross sections for xenon, including six stages of ionisation, give $B = 4{\cdot}37 \times 10^{-14}$ cm^2 V, $C = 0{\cdot}055$ V^{-1}. This value of C enables A to be interpolated for prescribed values of V_a and $J_-(0)/J_0$. The parameter na can then be determined from equations (11) and (14) using the above value of B. Figure 4 shows the results obtained for $V_a = 1$, 3, 10 and 30 kV where na has been expressed in terms of the product of gas pressure (Torr at 20 °C) and electrode spacing (mm).

Higher anode potentials are not considered due to the accuracy of the corrections that have been applied to take account of charge transfer collisions and relativistic effects. The average value of the cross section used for determining the probability of a charge transfer collision is that calculated for xenon by Rapp and Francis (1962), evaluated at half the anode potential. Over the range of variables covered in figure 4 this probability has a maximum value of $P_+ = 0{\cdot}11$, occurring at $V_a = 30$ kV and $J_-(0)/J_0 = 1{\cdot}7$. The analysis of § 2.3 indicates that the corresponding correction to the current ratio is +2%. An additional −1% correction at 30 kV arises from relativistic effects, given by equation (25).

According to equation (20) the maximum xenon ion density attained in the calculations is $n_+(0) = 1{\cdot}1 \times 10^{11}/a^2$ cm^{-3}. This is located at the cathode surface for $V_a = 30$ kV and $J_-(0)J_0 = 1{\cdot}7$, corresponding to $Y(0) = 0{\cdot}52$ in figure 3. The ion cross section σ_-^+ required in equation (19) has been measured for xenon by Latypov *et al*

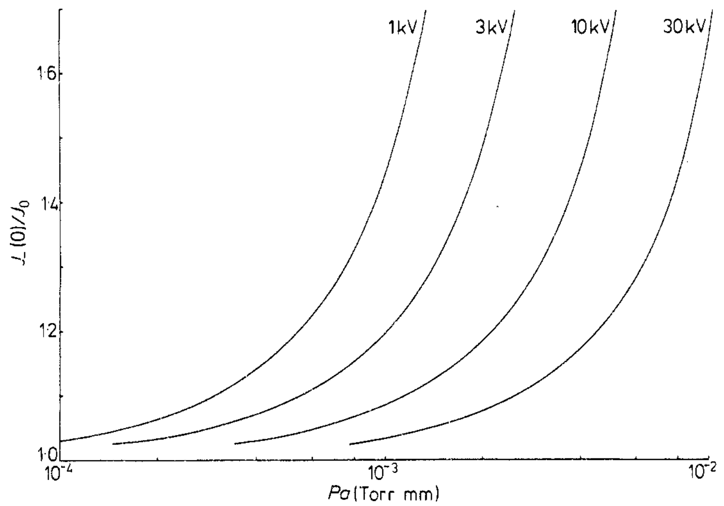

Figure 4. The electron current density as a function of the product of xenon filling pressure (Torr) and electrode spacing (mm) at selected values of anode potential.

(1964). They observed a peak value that was 17% greater than the peak value for neutral xenon. If this ratio of cross sections is maintained at all potentials then equation (19) indicates, with the above value of $n_+(0)$, that the increase in A at 30 kV due to electron–ion collisions is always less than $A \rightarrow (1+5\times10^{-3}/a)A$. At $J_-(0)/J_0 = 1{\cdot}7$ this fractional increase in A produces the same fractional increase in $J_-(0)/J_0$. Consequently, for an electrode spacing of $a = 1$ mm the neglect of electron–ion collisions produces an error in the current ratio that is less than 5%. At $V_a = 10$ kV the error is less than 2%.

For $V_a = 30$ kV and $J_-(0)/J_0 = 1{\cdot}7$ equation (26) gives $A \rightarrow (1+0{\cdot}03)A$. Consequently the neglect of the self magnetic field produces an error in the current ratio at 30 kV that is less than 3%. For an anode potential of 10 kV this error is less than 1%.

Table 1 gives the times τ_1 and τ_2 for pulsed operation of a xenon-filled diode according to equations (27) and (28). The values of τ_2 are appropriate to a maximum ionisation cross section of $4{\cdot}8\times10^{-16}$ cm^2, as given by Massey and Burhop (1969). For an electrode spacing of 1 mm and an anode potential in the vicinity of 3 kV the times τ_1 and τ_2 are comparable to the time scales associated with diode energisation by a strip transmission line. Such lines have a current rise time of about 20 ns, determined by

Table 1. The time scales τ_1, τ_2 for pulsed operation of a xenon-filled diode and the time τ_3 for establishment of the steady state at $J_-(0)/J_0 = 1{\cdot}7$.

V_a	$a = 1$ mm			$a = 1$ cm		
(kV)	τ_1 (ns)	τ_2 (ns)	τ_3 (μs)	τ_1 (μs)	τ_2 (μs)	τ_3 (μs)
1	39	530	1·5	0·39	53	150
3	23	100	0·66	0·23	10	66
10	12	17	0·29	0·12	1·7	29
30	7·2	3·2	0·14	0·072	0·32	14

switch characteristics, and a pulse length of about 100 ns, determined by the physical length of the line. For higher anode potentials, at this electrode spacing, the neutral depletion is too fast to permit a pulsed regime of operation. At the greater electrode spacing of 1 cm it is found that τ_2 is appreciably greater than τ_1 for all voltages presented and these time scales are well suited to diode energisation by a lumped parameter network. The value of f required in equation (31) is determined from tabulated values of atomic thermal velocity and mean free path. For xenon the value obtained is $f = 1{\cdot}20 \times 10^{18}\ \mathrm{cm}^{-1}\ \mathrm{s}^{-1}$ at 20 °C, inferred from the tables of Loeb (1934) at STP. For a current ratio $J_-(0)/J_0 = 1{\cdot}7$ equation (31) restricts the steady-state anode potential to $V_a \leqslant 4{\cdot}4$ kV for $a = 1$ mm and to $V_a \leqslant 22$ kV for $a = 1$ cm. The final column in table 1 presents the time scale τ_3 of equation (32) for the establishment of the steady state.

High-voltage diodes are frequently used as sources for electron beams and a low beam divergence is of prime importance. The inert gas atoms present a total scattering cross section for electrons that is between three and four times the ionisation cross section. However, for $J_-(0)/J_0 = 1{\cdot}7$ equation (21) indicates that a xenon gas filling causes less than 0·2% of the electron beam to suffer an ionising collision in the electrode gap. Consequently the total scatter in the beam at the anode, due to the gas filling, is less than 1%.

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